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Using maximality and minimality conditions to construct inequality chains

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Abstract

The following inequality chain has been extensively studied in the discrete mathematical literature:

$$\mathrm{ir} \leqslant \gamma \leqslant i \leqslant \beta \leqslant \Gamma \leqslant \mathrm{IR},$$

where ir and IR denote the lower and upper irredundance numbers of a graph, γ and Γ denote the lower and upper domination numbers of a graph, i denotes the independent domination number and β denotes the vertex independence number of a graph. More than one hundred papers have been published on aspects of this chain. In this paper we define a simple mechanism which explains why this inequality chain exists and how it is possible to define many similar chains of potentially arbitrary length.

1. Introduction

Graph theory terminology not presented here can be found in [6]. Let $G=(V,E)$ be a graph. For any vertex $v \in V$, the *open neighborhood of* v, denoted by $N(v)$, is defined by $\{u \in V \mid uv \in E\}$. The *closed neighborhood of* v, denoted by $N[v]$, is the set $N(v) \cup \{v\}$. For $S \subseteq V$, the *open neighborhood of* S, denoted by $N(S)$, is defined as $\bigcup_{v\in S} N(v)$, while the *closed neighborhood of* S, denoted by $N[S]$, is defined by $\bigcup_{v\in S} N[v]$. The *private neighbor set of a vertex* v *with respect to a set* S is denoted

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by $\mathrm{PN}[v,S] = N[v] - N[S - \{v\}]$. If $\mathrm{PN}[v,S] \neq \emptyset$ for some vertex v and some $S \subseteq V$, then every vertex of $\mathrm{PN}[v,S]$ is called a *private neighbor of v with respect to S*.

A set S is an *independent set* if no two vertices in S are adjacent; S is a *dominating set* if $N[S] = V$, or, equivalently, if for every vertex $u \in V - S$, there exists $v \in S$ such that $uv \in E$; and S is called an *irredundant set* if for every vertex $v \in S$, $\mathrm{PN}[v,S] \neq \emptyset$ (every vertex in S has a private neighbor with respect to S).

The *vertex independence number* of a graph G, denoted $\beta(G)$, is defined as $\max\{|S| \mid S$ is an independent set of $G\}$. The *independent domination number of* G, denoted $i(G)$, is defined as $\min\{|S| \mid S$ is a maximal independent set of $G\}$ (or, equivalently, as $\min\{|S| | S$ is an independent and dominating set of $G\}$.

The *domination number* of a graph G, denoted $\gamma(G)$, is defined as $\min\{|S| \mid S$ is a minimal dominating set of $G\}$, while the *upper domination number* of a graph G, denoted $\Gamma(G)$, is defined as $\max\{|S| \mid S$ is a minimal dominating set of $G\}$.

The *irredundance number* of a graph G, denoted $\mathrm{ir}(G)$, is defined as $\min\{|S| \mid S$ is a maximal irredundant set of $G\}$, while the *upper irredundance number* of a graph G, denoted $\mathrm{IR}(G)$, is defined as $\max\{|S| \mid S$ is a maximal irredundant set of $G\}$.

Notice first that from the definitions above, for any graph G,

$$i(G) \leqslant \beta(G), \qquad \gamma(G) \leqslant \Gamma(G) \quad \text{and} \quad \mathrm{ir}(G) \leqslant \mathrm{IR}(G).$$

But we can say more.

Proposition 1 (Berge [4]). *If G is a graph, then every maximal independent set is a minimal dominating set.*

Proof. To be discussed later. □

Proposition 2 (Cockayne et al. [11]). *If G is a graph, then every minimal dominating set is a maximal irredundant set.*

Proof. To be discussed later. □

By Propositions 1 and 2, we have

Corollary 1 (Cockayne et al. [11]). *If G is a graph, then*

$$\mathrm{ir}(G) \leqslant \gamma(G) \leqslant i(G) \leqslant \beta(G) \leqslant \Gamma(G) \leqslant \mathrm{IR}(G). \tag{1}$$

This inequality chain (1) first appeared in a paper by Cockayne et al. [11] in 1978. It has been the focus of more than 100 papers since then, including investigations such as the following:

1. Given an integer sequence $1 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant e \leqslant f$, does there exist a graph G for which $\mathrm{ir}(G) = a$, $\gamma(G) = b$, $i(G) = c$, $\beta(G) = d$, $\Gamma(G) = e$ and $\mathrm{IR}(G) = f$? This question was settled affirmatively by Cockayne and Mynhardt [14];

2. Under what conditions are any two parameters equal? For example,
 (a) $\gamma = i$ [2].
 (b) $i = \beta$ [18,19,42,44] (these are called well-covered graphs).
 (c) $\gamma = \Gamma$ [17] (these are called well-dominated graphs).
 (d) $\beta = \Gamma = \mathrm{IR}$ [7,36].
3. Are there other parameters of graphs whose values always lie between any two parameters in (1)? For example,
 (a) k-minimal and k-maximal independence parameters: $i \leqslant \beta_2 \leqslant \beta_3 \leqslant \cdots \leqslant \beta$ [12,13].
 (b) private domination: $\gamma \leqslant \Gamma_p \leqslant \Gamma$ [30].
4. Are there variants of the basic independence, domination and irredundance parameters that satisfy a similar inequality chain? For example,
 (a) edge versions: $\mathrm{ir}' \leqslant \gamma' = i' \leqslant \beta' \leqslant \Gamma' \leqslant \mathrm{IR}'$ [35,38] (Note that the edge domination number always equals the independent edge domination number.)
 (b) mixed (vertices and edges) versions: $\mathrm{ir}_m \leqslant \gamma_m \leqslant i_m \leqslant \beta_m \leqslant \Gamma_m \leqslant \mathrm{IR}_m$ [1,31].
 (c) fractional versions: $\mathrm{ir}_f \leqslant \gamma_f \leqslant \Gamma_f \leqslant \mathrm{IR}_f$ [15,21].
 (d) iterated versions: $\mathrm{ir}^* \leqslant \gamma^* \leqslant i^* \leqslant \beta^* \leqslant \Gamma^* \leqslant \mathrm{IR}^*$ [33].
 (e) multiple versions: $\mathrm{ir}_k \leqslant \gamma_k \leqslant i_k \leqslant \beta_k \leqslant \Gamma_k \leqslant \mathrm{IR}_k$ [16].
5. Are there parameters whose values are always smaller or always larger than those in (1)? For example,
 (a) external redundant sets: $\mathrm{er} \leqslant \mathrm{ir} \leqslant \cdots \leqslant \mathrm{IR} \leqslant \mathrm{ER}$ [39] (also to be discussed here).
 (b) irredundance variations: $\mathrm{OIR} \leqslant \mathrm{IR} \leqslant \mathrm{COIR}$ [20].
6. How do these parameters behave when restricted to various classes of graphs? For example,
 (a) chessboards [22].
 (b) grids [8].
 (c) hypercubes [25,27].

The following is a brief historical review of the development of the inequality chain in (1).

1. 1958 [3] Berge defines the coefficient of internal stability (which is β) and denotes it by α. Berge defines the coefficient of external stability (which is γ) and denotes it β.
2. 1962 [41] Ore introduces the domination number (which is γ) and denotes it δ.
3. 1969 [24] Harary introduces the notation β_0 for the independence number.
4. 1979 [9] Cockayne and Hedetniemi survey domination in graphs and define the parameters i and Γ and introduce the notation γ.
5. 1978 [11] Cockayne, Hedetniemi and Miller introduce irredundant sets, define ir and IR and exhibit (1) for the first time.

The paper is organized as follows. In Section 2 we define a simple mechanism which explains why this inequality chain exists and how it is possible to define many similar chains of potentially arbitrary length. In Section 3 we prove Gallai-type theorems for the parameters in the generalized inequality chain of Section 2. In Section 4 we indicate a further generalization and use hereditary families of subsets of an arbitrary set to generate such chains.

2. Generalized inequality chains for graphs

In this section we would like to show that there is a certain 'naturalness' to the inequality chain in (1) from which a greater degree of generality emerges.

Let $G=(V,E)$ be a graph. Let P be a property enjoyed by some of the subsets of V. A subset of V with (without) property P is called a P-set ($\overline{P}$-set). A property P is *hereditary* (*superhereditary*) if each subset (superset) of a P-set is also a P-set. Notice that the property of being an independent set is hereditary. The property of being an irredundant set can also be seen to be hereditary. Notice, furthermore, that the property of being a dominating set is superhereditary.

A subset S of V is called a *1-maximal P-set* of S if S has the property P, but $S\cup\{v\}$ is a $\overline{P}$-set for all $v\in V-S$. A subset S of V is called a *maximal P-set* if S has the property P, but for all proper supersets S' of S, S' is a $\overline{P}$-set. Clearly, maximal P-sets are always 1-maximal P-sets, but the converse is not always true. There are properties P and graphs G which contain P-sets that are 1-maximal but not maximal. One example is the property of being externally redundant (to be defined and illustrated later). However, we can assert the following.

Proposition 3. *Let $G=(V,E)$ be a graph and let P be a hereditary property. Then $S\subseteq V$ is a 1-maximal P-set if and only if S is a maximal P-set.*

Proof. By definition, every maximal P-set is 1-maximal.

For the converse, let S be a P-set of vertices which is 1-maximal. We want to show that S is maximal. Suppose, to the contrary, that S is not maximal. Then there exists a superset $S''\supset S$ which is a P-set, where $|S''|-|S|\geqslant 2$. But now, since the property P is hereditary, every subset of S'' is a P-set. In particular, every subset $S'\subset S''$ with $|S'|=|S|+1$ is a P-set. But this contradicts the assumption that S is 1-maximal, i.e., for every $v\in V-S$, $S\cup\{v\}$ is $\overline{P}$-set. □

A similar situation holds for minimal P-sets. A subset S of V is called a *1-minimal P-set* if S has the property P, but $S-\{v\}$ is a $\overline{P}$-set for all $v\in S$. A subset S of V is called a *minimal P-set* if S has the property P, but for all proper subsets S' of S, S' is a $\overline{P}$-set. Clearly, minimal P-sets are 1-minimal P-sets, but the converse is not always true. Using the same type of argument as in Proposition 3, we can assert the following:

Proposition 4. *Let $G=(V,E)$ be a graph and let P be a superhereditary property. Then the set $S\subseteq V$ is a 1-minimal P-set if and only if S is a minimal P-set.*

Let $G=(V,E)$ be a graph and let $\mathscr{H}$ be a set of graphs. For $u,v\in V$ and $S\subseteq V$, we say that u and v are *$\mathscr{H}$-adjacent in S* if there exists an $H\in\mathscr{H}$ such that $\langle S\rangle$ contains a copy of H containing u and v. Let $S\subseteq V$ be a set for which $|S|\geqslant \min\{p(H)-1 \mid H\in \mathscr{H}\}$ and $v\in S$. The *open $\mathscr{H}$-neighborhood of v with respect to S*, denoted by $N^S_{\mathscr{H}}(v)$, is defined as $\{u\in V \mid u$ and v are $\mathscr{H}$-adjacent in $S\cup\{u\}\}$. The *closed $\mathscr{H}$-neighborhood of v with respect to S*, denoted by $N^S_{\mathscr{H}}[v]$, is defined as $N^S_{\mathscr{H}}(v)\cup\{v\}$. Also, let $N^S_{\mathscr{H}}[S] = \bigcup_{v\in S} N^S_{\mathscr{H}}[v]$.

We now discuss the 'naturalness' inherent in the inequality chain of (1). In particular, we will see that this inequality chain is 'grown' from the property P of being an independent set; from this every other property 'naturally' follows by alternately defining maximality and minimality conditions of successive types of sets. To avoid repetition, we look at the inequality chain at a slightly higher level. (The classic inequality chain will then materialize if we set $\mathscr{H}=\{K_2\}$.)

Let P_0 be the following *seed* property: a set $S\subseteq V$ has property P_0 if and only if for all $H\in\mathscr{H}, \langle S\rangle$ contains no H, i.e., S is *$\mathscr{H}$-independent* or *$\mathscr{H}$-free.*

Step 1.1. Let P_1 be the property that is obtained by characterizing P_0-sets which are maximal. Since the property of being an $\mathscr{H}$-independent set is hereditary, Proposition 3 implies that we can just as well characterize 1-maximal $\mathscr{H}$-independent sets. Thus, an $\mathscr{H}$-independent set S is maximal if and only if for every $v\in V-S$, $S\cup\{v\}$ is not $\mathscr{H}$-independent. But this is equivalent to the following:

Maximality condition for $\mathscr{H}$-independent sets:

P_1: for every $v\in V-S$, $\langle S\cup\{v\}\rangle$ contains an $H\in\mathscr{H}$ containing v, i.e., $v\in N^S_{\mathscr{H}}[S]$.

If a set has property P_1, we will call it an *$\mathscr{H}$-dominating set.*

Step 1.2. We now prove the following result: if $S\subseteq V$ is a maximal P_0-set, then S is a minimal P_1-set, or, equivalently,

Proposition 5. *If $S\subseteq V$ is a maximal $\mathscr{H}$-independent set, then S is a minimal $\mathscr{H}$-dominating set of G.*

Proof. Since the property of being an $\mathscr{H}$-dominating set is a superhereditary property, Proposition 4 implies that the notions of minimal and 1-minimal coincide.

Let S be a maximal $\mathscr{H}$-independent set. Since S is maximal, S is an $\mathscr{H}$-dominating set. We now prove that S is a *minimal* $\mathscr{H}$-dominating set. Suppose not. Then there exists a $v\in S$ such that $S-\{v\}$ is an $\mathscr{H}$-dominating set. Since $S-\{v\}$ is an $\mathscr{H}$-dominating set, $\langle S-\{v\}\cup\{v\}\rangle=\langle S\rangle$ contains an $H\in\mathscr{H}$ containing v, contradicting the fact that S is $\mathscr{H}$-independent. □

Note that if $\mathscr{H}=\{K_2\}$, then $\mathscr{H}$-independence is independence, $\mathscr{H}$-domination is domination, while Proposition 5 generalizes Proposition 1.

Let $i_{\mathscr{H}}$ ($\beta_{\mathscr{H}}$) and $\gamma_{\mathscr{H}}$ ($\Gamma_{\mathscr{H}}$) be the minimum (maximum) cardinality of a maximal $\mathscr{H}$-independent (P_0) set and a minimal $\mathscr{H}$-dominating (P_1) set. Proposition 5 implies that $\gamma_{\mathscr{H}}\leqslant i_{\mathscr{H}}$ and $\beta_{\mathscr{H}}\leqslant\Gamma_{\mathscr{H}}$.

Let us now repeat Steps 1.1 and 1.2 for property P_1, i.e., for $\mathscr{H}$-dominating sets.

Step 2.1. Let P_2 be the property that is obtained by characterizing P_1-sets which are minimal. Since the property of being an $\mathscr{H}$-dominating set is superhereditary, Proposition 4 implies that we can just as well characterize 1-minimal $\mathscr{H}$-dominating sets. Thus, an $\mathscr{H}$-dominating set S is minimal if and only if for every $v \in S$, $S - \{v\}$ is not $\mathscr{H}$-dominating if and only if for every $v \in S$, there exists a $w \in V - (S - \{v\})$ such that $w \notin N_{\mathscr{H}}^{S-\{v\}}[S - \{v\}]$. However, since the latter condition is the minimality condition for the *$\mathscr{H}$-dominating set S* and since $w \notin N_{\mathscr{H}}^{S-\{v\}}[S - \{v\}]$, we must have that $w \in N_{\mathscr{H}}^{S}[v]$. Hence $w \in N_{\mathscr{H}}^{S}[v] - N_{\mathscr{H}}^{S-\{v\}}[S - \{v\}]$. Conversely, if $w \in V - (S - \{v\})$ is such that that $w \in N_{\mathscr{H}}^{S}[v] - N_{\mathscr{H}}^{S-\{v\}}[S - \{v\}]$, then $w \notin N_{\mathscr{H}}^{S-\{v\}}[S - \{v\}]$.

Hence, we have the following:

Minimality condition for $\mathscr{H}$-dominating sets:

P_2: for every $v \in S$, $\mathrm{PN}_{\mathscr{H}}[v, S] := N_{\mathscr{H}}^{S}[v] - N_{\mathscr{H}}^{S-\{v\}}[S - \{v\}] \neq \emptyset$.

If a set S has property P_2, we will call it an *$\mathscr{H}$-irredundant set.*

Step 2.2. We now prove the following result: if $S \subseteq V$ is a minimal P_1-set, then S is a maximal P_2-set, or, equivalently,

Proposition 6. *If $S \subseteq V$ is a minimal $\mathscr{H}$-dominating set, then S is a maximal $\mathscr{H}$-irredundant set of G.*

Proof. Let S be a minimal $\mathscr{H}$-dominating set. Since S is a *minimal* $\mathscr{H}$-dominating set, it is also an $\mathscr{H}$-irredundant set. We now prove that S is a *maximal* $\mathscr{H}$-irredundant set. Suppose not. Then there exists $S' \supset S$ such that S' is an $\mathscr{H}$-irredundant set. Let $v \in S' - S$. Then, since $v \notin S$ and S is an $\mathscr{H}$-dominating set, there exists an $H \in \mathscr{H}$ such that $\langle S \cup \{v\} \rangle$ contains a subgraph isomorphic to H containing v. Now, since $S \cup \{v\} \subseteq S'$, $\langle S' \rangle$ contains a subgraph isomorphic to H containing v. Since S' is an $\mathscr{H}$-irredundant set, this implies that there exists $w \in V - S'$ such that $w \in \mathrm{PN}_{\mathscr{H}}[v, S']$. Also, since $V - S' \subseteq V - S$, it follows that $w \in V - S$. Since S is an $\mathscr{H}$-dominating set, $\langle S \cup \{w\} \rangle$ contains an $H' \in \mathscr{H}$ containing w. But $S \cup \{w\} \subseteq S' \cup \{w\} - \{v\}$, so that $\langle S' \cup \{w\} - \{v\} \rangle$ contains $H' \in \mathscr{H}$ containing w, which implies that $w \notin \mathrm{PN}_{\mathscr{H}}[v, S']$, a contradiction. □

Let $\mathrm{ir}_{\mathscr{H}}$ ($\mathrm{IR}_{\mathscr{H}}$) be the minimum (maximum) cardinality of a maximal $\mathscr{H}$-irredundant (P_2) set. Proposition 6 implies that $\mathrm{ir}_{\mathscr{H}} \leqslant \gamma_{\mathscr{H}}$, while $\Gamma_{\mathscr{H}} \leqslant \mathrm{IR}_{\mathscr{H}}$.

Note that if $\mathscr{H} = \{K_2\}$, then $\mathscr{H}$-irredundance is irredundance, while Proposition 6 generalizes Proposition 2.

It is easily verified that the set $S = \{1, 2, 3, 4, 5\}$ of the graph of Fig. 1 is a $\{K_3\}$-irredundant set, while $S - \{1\}$ is not a $\{K_3\}$-irredundant set. Hence, the property of $\mathscr{H}$-irredundance is not, in general, hereditary. Note, however, that irredundance is hereditary.

We now repeat Steps 2.1 and 2.2 for property P_2, i.e., for $\mathscr{H}$-irredundant sets.

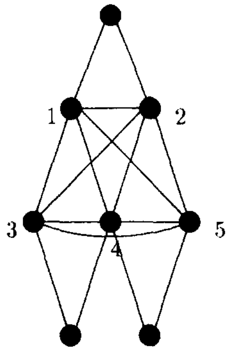

Fig. 1. A graph illustrating that $\{K_3\}$-irredundance is not hereditary.

Step 3.1. Let P_3 be the property that is obtained by characterizing P_2-sets which are maximal. An $\mathscr{H}$-irredundant set S is maximal if and only if for every $S' \supset S$, S' is not an $\mathscr{H}$-irredundant set if and only if for every $S' \supset S$, there exists $v \in S'$ such that $\mathrm{PN}_{\mathscr{H}}[v, S'] = \emptyset$. However, since the latter condition is the maximality condition for the *$\mathscr{H}$-irredundant set* S, if $v \in S$, then $\mathrm{PN}_{\mathscr{H}}[v, S] \neq \emptyset$. Conversely, if for every $S' \supset S$, there exists $v \in S'$ such that $\mathrm{PN}_{\mathscr{H}}[v, S'] = \emptyset$ and if $v \in S$, then $\mathrm{PN}_{\mathscr{H}}[v, S] \neq \emptyset$, then for every $S' \supset S$, there exists $v \in S'$ such that $\mathrm{PN}_{\mathscr{H}}[v, S'] = \emptyset$.

Hence, we have the following:

Maximality condition for $\mathscr{H}$-irredundant set:

P_3: for every $S' \supset S$, there exists an $v \in S'$ such that $\mathrm{PN}_{\mathscr{H}}[v, S'] = \emptyset$ and if $v \in S$, then $\mathrm{PN}_{\mathscr{H}}[v, S] \neq \emptyset$.

If a set S has property P_3, we will call it an *$\mathscr{H}$-external redundant set*.

Step 3.2. We now prove the following result: if $S \subseteq V$ is a maximal P_2-set, then S is a minimal P_3-set, or, equivalently,

Proposition 7. *If $S \subseteq V$ is a maximal $\mathscr{H}$-irredundant set, then S is a minimal $\mathscr{H}$-external redundant set of G.*

Proof. Let S be a maximal $\mathscr{H}$-irredundant set. Since S is a *maximal* $\mathscr{H}$-irredundant set, it is also an $\mathscr{H}$-external redundant set. We now prove that S is a *minimal* $\mathscr{H}$-external redundant set. Suppose not. Then there exists $S' \subset S$ such that S' is $\mathscr{H}$-external redundant. Since $S' \subset S$ and S' is $\mathscr{H}$-external redundant, there exists a $v \in S$ such that $\mathrm{PN}_{\mathscr{H}}[v, S] = \emptyset$ and if $v \in S'$, then $\mathrm{PN}_{\mathscr{H}}[v, S'] \neq \emptyset$. Since S is an $\mathscr{H}$-irredundant set, $\mathrm{PN}_{\mathscr{H}}[v, S] \neq \emptyset$, which is a contradiction. $\square$

Note that if $\mathscr{H} = \{K_2\}$, then external $\mathscr{H}$-redundance is external redundance, first defined by McRae [39]. We say that a set S is an *external redundant set* if for every vertex $v \in V - S$ there exists $w \in S \cup \{v\}$ such that $\mathrm{PN}[w, S \cup \{v\}] = \emptyset$ and if $w \in S$, then $\mathrm{PN}[w, S] \neq \emptyset$. To see that the property of $\mathscr{H}$-external redundance is not, in general

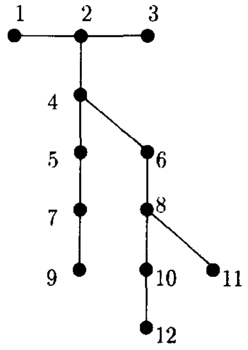

Fig. 2. A graph illustrating that external redundance is not superhereditary.

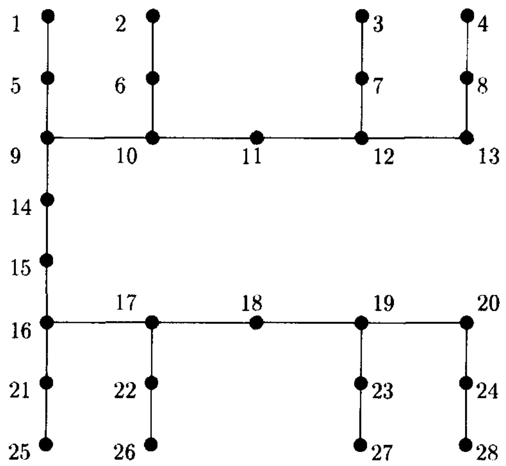

Fig. 3. The graph G_1 illustrates that $\mathrm{er}(G_1) \leqslant 9 < 10 = \mathrm{ir}(G_1)$.

a superhereditary property, consider the graph of Fig. 2. It is easily verified that the set $S = \{1,3,4,5,8,11\}$ is a 1-minimal external redundant set, while $S - \{1,3\}$ is an external redundant set, so that S is not minimal. Therefore, by Proposition 4, external redundance is not superhereditary.

Let $\mathrm{er}_{\mathscr{H}}$ ($\mathrm{ER}_{\mathscr{H}}$) be the minimum (maximum) cardinality of a minimal $\mathscr{H}$-external redundant (P_2) set. Proposition 7 implies that $\mathrm{er}_{\mathscr{H}} \leqslant \mathrm{ir}_{\mathscr{H}}$, while $\mathrm{IR}_{\mathscr{H}} \leqslant \mathrm{ER}_{\mathscr{H}}$. Since $i_{\mathscr{H}} \leqslant \beta_{\mathscr{H}}$, these results can be summarized as

Theorem 1. *If G is a graph and $\mathscr{H}$ is a set of graphs, then*

$$\mathrm{er}_{\mathscr{H}}(G) \leqslant \mathrm{ir}_{\mathscr{H}}(G) \leqslant \gamma_{\mathscr{H}}(G) \leqslant i_{\mathscr{H}}(G) \leqslant \beta_{\mathscr{H}}(G) \leqslant \Gamma_{\mathscr{H}}(G) \leqslant \mathrm{IR}_{\mathscr{H}}(G) \leqslant \mathrm{ER}_{\mathscr{H}}(G). \quad (2)$$

The vertex set $\{9,10,12,13,15,17,19,20\}$ in the graph G_1 of Fig. 3 is an external redundant set, showing that $\mathrm{er}(G_1) \leqslant 9$. Also, since G_1 is a tree, the linear algorithm of Bern et al. [5] for computing $\mathrm{ir}(T)$ of an arbitrary tree T can be used to show that

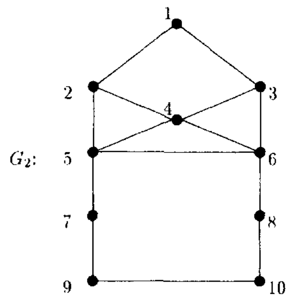

Fig. 4. The graph G_2 illustrates that $\mathrm{IR}(G_2) = 4 < 5 \leqslant \mathrm{ER}(G_2)$.

$\mathrm{ir}(G_1) = 10$. Hence, $\mathrm{er}(G_1) < \mathrm{ir}(G_1)$ is possible. For the graph G_2 of Fig. 4, it is easy to verify that $\mathrm{IR}(G_2) = 4$. Also, since the vertex set $\{1, 7–9, 10\}$ is a minimal external redundant set of G_2, we have that $\mathrm{ER}(G_2) \geqslant 5$. This shows that $\mathrm{IR}(G_2) < \mathrm{ER}(G_2)$ can also occur.

Also, we have the following result:

Proposition 8. *If $S \subseteq V$, then*

(a) *S is a maximal $\mathscr{H}$-independent set if and only if S is $\mathscr{H}$-independent and $\mathscr{H}$-dominating,*

(b) *S is a minimal $\mathscr{H}$-dominating set if and only if S is $\mathscr{H}$-dominating and $\mathscr{H}$-irredundant, and*

(c) *S is a maximal $\mathscr{H}$-irredundant set if and only if S is $\mathscr{H}$-irredundant and $\mathscr{H}$-external redundant.*

Proof. The proof is similar to the proofs of Propositions 5–7. □

The approach followed here, led to the discovery of a new concept: that of $\mathscr{H}$-external redundance. Notice that the inequality chain of (1) is a special case of the following general procedure: We start with a seed property P_0 and consider the property P_1 which is obtained by characterizing sets which are maximal with respect to P_0. If the statement 'If S is a maximal P_0-set, then S is a minimal P_1-set.' is true, continue by considering the property P_2 obtained by characterizing sets which are minimal with respect to P_1. If the statement 'If S is a minimal P_1-set, then S is a maximal P_2-set.' is true, continue by considering the property P_3 obtained by characterizing sets which are maximal with respect to P_2. etc. Let $\pi_0(\Pi_0)$ be the minimum (maximum) cardinality of a maximal P_0-set and let π_1 (Π_1) be the minimum (maximum) cardinality of a minimal P_1-set. Etc. Then $\cdots \leqslant \pi_1 \leqslant \pi_0 \leqslant \Pi_0 \leqslant \Pi_1 \leqslant \cdots$.

We pause briefly to discuss another generalization of domination (see, e.g., [28, 29, 34]) and show that these concepts differ.

Let $G = (V, E)$ be graph and let $\mathscr{H}$ be a set of graphs. Two distinct vertices of the graph G are said to be *$\mathscr{H}$-adjacent* if they are contained in a subgraph of G which is isomorphic to a member of $\mathscr{H}$. The *open $\mathscr{H}$-neighborhood of a vertex v* of G, denoted $N_{\mathscr{H}}(v)$, is defined as $\{u \in V(G) | u$ and v are $\mathscr{H}$-adjacent$\}$ and the *closed $\mathscr{H}$-neighborhood of a vertex v*, denoted $N_{\mathscr{H}}[v]$, is defined as $N_{\mathscr{H}}(v) \cup \{v\}$. The *closed $\mathscr{H}$-neighborhood of a set $S \subseteq V$* is defined as $\bigcup_{v \in S} N_{\mathscr{H}}[v]$. A set $S \subseteq V$ is an *$\mathscr{H}$-dominating set* if $V(G) = N_{\mathscr{H}}[S]$. An $\mathscr{H}$-dominating set S is called an *$\mathscr{H}$-independent dominating set* of G if no two vertices in S are $\mathscr{H}$-adjacent in G. A set $S \subseteq V$ is called an *$\mathscr{H}$-irredundant set* if $N_{\mathscr{H}}[v] - N_{\mathscr{H}}[S - \{v\}] \neq \emptyset$ for all $v \in S$. Let $i_{\mathscr{H}}$ ($\beta_{\mathscr{H}}$), $\gamma_{\mathscr{H}}$ ($\Gamma_{\mathscr{H}}$) and $\mathrm{ir}_{\mathscr{H}}$ ($\mathrm{IR}_{\mathscr{H}}$) be the minimum (maximum) cardinality of a maximal $\mathscr{H}$-independent set, a minimal $\mathscr{H}$-dominating set and a maximal $\mathscr{H}$-irredundant set.

The *$\mathscr{H}$-power graph of G*, denoted $G_{\mathscr{H}}$, is the graph with vertex set $V(G)$ and edge set $\{uv | u$ and v are $\mathscr{H}$-adjacent in $G\}$. The following result is easy to prove.

Theorem 2. *If G is a graph and $\mathscr{H}$ is a set of graphs, then*

$$\mathrm{ir}(G_{\mathscr{H}}) = \mathrm{ir}_{\mathscr{H}}(G), \tag{3}$$

$$i(G_{\mathscr{H}}) = i_{\mathscr{H}}(G), \tag{4}$$

$$\gamma(G_{\mathscr{H}}) = \gamma_{\mathscr{H}}(G), \tag{5}$$

$$\beta(G_{\mathscr{H}}) = \beta_{\mathscr{H}}(G), \tag{6}$$

$$\Gamma(G_{\mathscr{H}}) = \Gamma_{\mathscr{H}}(G), \tag{7}$$

$$\mathrm{IR}(G_{\mathscr{H}}) = \mathrm{IR}_{\mathscr{H}}(G). \tag{8}$$

These generalized parameters differ from our generalized parameters. For example, let $G = K_2 \times K_3$. Then $\gamma_{K_3}(G) = \gamma(G_{\{K_3\}}) = 2$, in the context of Theorem 8. However, as defined in Step 1.1, γ_{K_3} is equal to 4 for the graph G. Note that for $\mathscr{H} = \{K_2\}$, however, these parameters are the same.

3. Generalized Gallai theorems for inequality chains

We now consider yet another application of our procedure for generating inequality chains which will show that natural Gallai theorems exist for the parameters in the generalized inequality chain of the previous section.

We begin with a brief historical account. Let $\alpha(G)$ denote the vertex covering number of a graph $G = (V, E)$, i.e., the minimum cardinality of a set $S \subseteq V$ such that for every edge $uv \in E$, $u \in S$ or $v \in S$. Gallai [23] presented the following, now classical, result:

Theorem 3. *For any graph G of order p,* $\alpha(G) + \beta(G) = p$.

In [37], McFall and Nowakowski, generalized Gallai's result. Let $\alpha^+(G)$ denote the maximum cardinality of a minimal vertex cover.

Theorem 4. *For any graph G of order p,* $\alpha^+(G) + i(G) = p$.

Nieminen [40] suggested the following result. Let $\varepsilon_F(G)$ denote the maximum number of pendant edges in a spanning forest of G.

Theorem 5. *For any graph G of order p, $\gamma(G)+\varepsilon_F(G)=p$.*

Goldman, in a paper by Slater (see [43]), defines a vertex v to be an *enclave of* $S\subseteq V$ if $N[v]\subseteq S$. An *enclaveless set* is one that does not contain any enclave, that is, each $N[v]$ in G contains at most $\deg(v)$ elements of S. Let $\psi(G)$ denote the minimum cardinality of a maximal enclaveless set. The restriction of Theorem 2 in [43] from hypergraphs to graphs gives the following Gallai-type result.

Theorem 6. *For any graph of order p, $\Gamma(G)+\psi(G)=p$.*

Many other results like these, now called Gallai theorems, were subsequently found by Hedetniemi [32] and Cockayne et al. [10].

We now consider yet another application of our procedure for generating parameter inequality chains.

Let $G=(V,E)$ be a graph and let $\mathscr{H}$ be a set of nontrivial subgraphs of G. Let $\mathscr{E}_{\mathscr{H}}(G)=\{H'\subseteq G\mid$ there exists $H\in\mathscr{H}$ such that $H'\cong H\}$ and let $N_{\mathscr{H}}(v)=\{H'\subseteq G\mid$ there exists $H\in\mathscr{H}$ such that $H'\cong H$ and $v\in V(H')\}$. For a set $S\subseteq V$, define $N_{\mathscr{H}}(S)=\bigcup_{v\in S}N_{\mathscr{H}}(v)$. We say that a set $S\subseteq V$ is a *vertex-$\mathscr{H}$-dominating set* or a *$v\mathscr{H}$-dominating set* of G if $N_{\mathscr{H}}(S)=\mathscr{E}_{\mathscr{H}}(G)$.

Let P_0 be the following *seed* property: a set $S\subseteq V$ has property P_0 if and only if S is a $v\mathscr{H}$-dominating set of G. Let P_1 be the property that is obtained by characterizing P_0-sets which are minimal. Since the property of being a $v\mathscr{H}$-dominating set is superhereditary, we may just as well characterize 1-minimal $v\mathscr{H}$-dominating sets. Thus, a $v\mathscr{H}$-dominating set is minimal if and only if for every $v\in S$, $S-\{v\}$ is not $v\mathscr{H}$-dominating if and only if for every $v\in S$, there exists $H'\subseteq G$ and $H\in\mathscr{H}$ such that $H'\cong H$ and $V(H')\cap(S-\{v\})=\emptyset$. For $S\subseteq V$ and $v\in S$, let $p\mathscr{H}\ (v,S)=\{H'\subseteq G\mid$ there exists $H\in\mathscr{H}$ such that $H'\cong H$, $V(H')\cap(S-\{v\})=\emptyset$ and $v\in V(H')\}$. Then, equivalently, a set $S\subseteq V$ has property P_1 if and only if for every $v\in S$, $p\mathscr{H}\ (v,S)\neq\emptyset$. If a set has property P_1 we will call it a *$v\mathscr{H}$-irredundant set.*

Proposition 9. *If $S\subseteq V$ is a minimal $v\mathscr{H}$-dominating set of G, then S is a maximal $v\mathscr{H}$-irredundant set.*

Proof. Since the property of being a $v\mathscr{H}$-irredundant set is a hereditary property, Proposition 3 implies that the notions of maximal and 1-maximal coincide.

Let S be a minimal $v\mathscr{H}$-dominating set. Since S is minimal, S is also a $v\mathscr{H}$-irredundant set. We now prove that S is a *maximal* $v\mathscr{H}$-irredundant set. Suppose not. Then there exists $v\in V-S$ such that $S\cup\{v\}$ is $v\mathscr{H}$-irredundant. Since $S\cup\{v\}$ is $v\mathscr{H}$-irredundant, $p\mathscr{H}\ (v,S\cup\{v\})\neq\emptyset$. Hence, there exists $H'\subseteq G$ and $H\in\mathscr{H}$ such that $H' \cong H$, $V(H') \cap S = \emptyset$ and $v \in V(H')$. However, since S is a $v\mathscr{H}$-dominating set, $V(H') \cap S \neq \emptyset$. This contradiction establishes the result. $\square$

Let $\gamma'_{\mathscr{H}}(\Gamma'_{\mathscr{H}})$ and $\mathrm{ir}'_{\mathscr{H}}(\mathrm{IR}'_{\mathscr{H}})$ be the minimum (maximum) cardinality of a minimal $v\mathscr{H}$-dominating (P_0) set and a maximal $v\mathscr{H}$-irredundant (P_1) set. Proposition 9 implies that $\mathrm{ir}'_{\mathscr{H}} \leqslant \gamma'_{\mathscr{H}}$ and $\Gamma'_{\mathscr{H}} \leqslant \mathrm{IR}'_{\mathscr{H}}$. Note that, if $\mathscr{H} = \{K_2\}$, then $\gamma'_{\mathscr{H}} = \alpha$, $\Gamma'_{\mathscr{H}} = \alpha^+$, $\mathrm{ir}'_{\mathscr{H}} = \psi$, while $\mathrm{IR}'_{\mathscr{H}} = \varepsilon_F$.

Let P_2 be the property that is obtained by characterizing P_1-sets S which are maximal. Since the property of being a $v\mathscr{H}$-irredundant set is hereditary, we may just as well characterize 1-maximal $v\mathscr{H}$-irredundant sets. Thus, a $v\mathscr{H}$-irredundant set is maximal if and only if for every $v \in V - S$, $S \cup \{v\}$ is not a $v\mathscr{H}$-irredundant set if and only if for every $v \in V - S$, there exists $w \in S \cup \{v\}$ such that $p\mathscr{H}\ (w, S \cup \{v\}) = \emptyset$. However, since S is *$v\mathscr{H}$-irredundant*, if $w \in S$, then $p\mathscr{H}\ (w, S) \neq \emptyset$. Thus, if for every $v \in V - S$, there exists $w \in S \cup \{v\}$ such that $p\mathscr{H}\ (w, S \cup \{v\}) = \emptyset$ and if $w \in S$, then $p\mathscr{H}\ (w, S) \neq \emptyset$.

Hence, the maximality condition for $v\mathscr{H}$-irredundant sets is equivalent to:

P_2: for every $v \in V - S$, there exists $w \in S \cup \{v\}$ such that $p\mathscr{H}\ (w, S \cup \{v\}) = \emptyset$ and if $w \in S$, then $p\mathscr{H}\ (w, S) \neq \emptyset$.

If a set S has property P_2, we will call it a *$v\mathscr{H}$-external redundant set.*

Proposition 10. *If $S \subseteq V$ is a maximal $v\mathscr{H}$-irredundant set, then S is a minimal $v\mathscr{H}$-external redundant set.*

Proof. Let S be a maximal $v\mathscr{H}$-irredundant set. Since S is a *maximal* $v\mathscr{H}$-irredundant set, it is also a $v\mathscr{H}$-external redundant set. We now prove that S is a *minimal $v\mathscr{H}$*-redundant set. Suppose not. Then there exists $S' \subset S$ such that S' is a $v\mathscr{H}$-external redundant set. Let $v \in S - S'$. Then $v \in V - S'$ and since S' is $v\mathscr{H}$-external redundant, there exists $w \in S' \cup \{v\}$ such that $p\mathscr{H}\ (w, S' \cup \{v\}) = \emptyset$ and if $w \in S'$, then $p\mathscr{H}\ (w, S') \neq \emptyset$. Since S is a $v\mathscr{H}$-irredundant set and $w \in S$, it follows that $p\mathscr{H}\ (w, S) \neq \emptyset$, i.e., there exists $H' \subseteq G$ and $H \in \mathscr{H}$ such that $H' \cong H$, $V(H') \cap (S - \{w\}) = \emptyset$ and $w \in V(H')$. Since $v \in S - S'$, we have that $S' \cup \{v\} \subseteq S$. This, and the fact that $V(H') \cap (S - \{w\}) = \emptyset$, imply that $V(H') \cap (S' \cup \{v\} - \{w\}) = \emptyset$, so that $H' \in p\mathscr{H}\ (w, S' \cup \{v\})$, which is a contradiction. $\square$

Let $\mathrm{er}'_{\mathscr{H}}(\mathrm{ER}'_{\mathscr{H}})$ be the minimum (maximum) cardinality of a minimal $v\mathscr{H}$-external redundant set. Proposition 10 implies that $\mathrm{er}'_{\mathscr{H}} \leqslant \mathrm{ir}'_{\mathscr{H}}$ and $\mathrm{IR}'_{\mathscr{H}} \leqslant \mathrm{ER}'_{\mathscr{H}}$. Since $\gamma'_{\mathscr{H}} \leqslant \Gamma'_{\mathscr{H}}$, these results can be summarized as

Theorem 7. *If G is a graph and $\mathscr{H}$ is a set of nontrivial subgraphs of G, then*

$$\mathrm{er}'_{\mathscr{H}} \leqslant \mathrm{ir}'_{\mathscr{H}} \leqslant \gamma'_{\mathscr{H}} \leqslant \Gamma'_{\mathscr{H}} \leqslant \mathrm{IR}'_{\mathscr{H}} \leqslant \mathrm{ER}'_{\mathscr{H}}.$$

In order to present our generalized Gallai theorems, we prove a few preliminary results.

Lemma 1. *Let $G=(V,E)$ be a graph and $\mathscr{H}$ be a set of nontrivial subgraphs of G. Then $S \subseteq V$ is an $\mathscr{H}$-independent set if and only if $V-S$ is a $v\mathscr{H}$-dominating set.*

Proof. Let S be an $\mathscr{H}$-independent set and let $S' = V-S$. Then S' is a $v\mathscr{H}$-dominating set, for otherwise there exists $H' \subseteq G$ and $H \in \mathscr{H}$ such that $H' \cong H$ and $V(H') \subseteq S$. It now follows that $H' \subseteq \langle S \rangle$, contradicting the fact that S is an $\mathscr{H}$-independent set.

Conversely, suppose that $S' = V-S$ is a $v\mathscr{H}$-dominating set. Then, since $S \cap S' = \emptyset$, S must be an $\mathscr{H}$-independent set. $\square$

Lemma 2. *Let $G=(V,E)$ be a graph and $\mathscr{H}$ be a set of nontrivial subgraphs of G. Then $S \subseteq V$ is a $v\mathscr{H}$-irredundant set if and only if $V-S$ is an $\mathscr{H}$-dominating set.*

Proof. Let S be a $v\mathscr{H}$-irredundant set and let $S' = V-S$. We now show that S' is an $\mathscr{H}$-dominating set. Let $v \in S$. Since $p\mathscr{H}\ (v,S) \neq \emptyset$, there exists $H' \subseteq G$ and $H \in \mathscr{H}$ such that $H' \cong H$, $V(H') \cap (S - \{v\}) = \emptyset$ and $v \in V(H')$. Hence, there exists $H' \subseteq G$ and $H \in \mathscr{H}$ such that $H' \cong H$, $H' \subseteq \langle S' \cup \{v\} \rangle$ with H' containing v. This implies that v is $\mathscr{H}$-dominated by S'.

For the converse, let S be an $\mathscr{H}$-dominating set and let $S' = V-S$. We show that S' is a $v\mathscr{H}$-irredundant set. Let $v \in S'$. Since $v \in S'$ and S is an $\mathscr{H}$-dominating set, there exists $H' \subseteq G$ and $H \in \mathscr{H}$ such that $H' \cong H$ with $\langle S \cup \{v\} \rangle$ containing H' containing v. But then $V(H') \cap (S' - \{v\}) = \emptyset$, so that $H' \in p\mathscr{H}\ (v,S')$. Since v was chosen arbitrarily, this implies that S' is a $v\mathscr{H}$-irredundant set. $\square$

Lemma 3. *Let $G=(V,E)$ be a graph and $\mathscr{H}$ be a set of nontrivial subgraphs of G. Then $S \subseteq V$ is an $\mathscr{H}$-irredundant set if and if $V-S$ is a $v\mathscr{H}$-external redundant set.*

Proof. Let S be an $\mathscr{H}$-irredundant set and let $S' = V-S$. We show that S' is a $v\mathscr{H}$-external redundant set. Let $v \in S$. Since S is an $\mathscr{H}$-irredundant set, $\mathrm{PN}_{\mathscr{H}}[v,S] \neq \emptyset$. Suppose first that $\mathrm{PN}_{\mathscr{H}}[v,S] \subseteq S$. We show that $p\mathscr{H}\ (v, S' \cup \{v\}) = \emptyset$. Suppose, to the contrary, that there exists $H' \subseteq G$ and $H \in \mathscr{H}$ such that $H' \cong H$, $V(H') \cap S' = \emptyset$ and $v \in V(H')$. Hence, $V(H') \subseteq S$. Let $w \in \mathrm{PN}_{\mathscr{H}}[v,S]$. If $w \neq v$, then, since $\mathrm{PN}_{\mathscr{H}}[v,S] \subseteq S$, it follows that $w \in S - \{v\}$. This implies that $w \in N_{\mathscr{H}}^{S-\{v\}}[S-\{v\}]$. This contradiction shows that $w = v$. Let $x \in V(H') - \{v\}$. Then $v \in N_{\mathscr{H}}^{S-\{v\}}[S-\{v\}]$, so that $v \notin \mathrm{PN}_{\mathscr{H}}[v,S]$. This final contradiction shows that $p\mathscr{H}\ (v, S' \cup \{v\}) = \emptyset$. We now assume that $\mathrm{PN}_{\mathscr{H}}[v,S] \cap S' \neq \emptyset$ and let $w \in \mathrm{PN}_{\mathscr{H}}[v,S] \cap S'$. Then there exists $H' \subseteq G$ and $H_1 \in \mathscr{H}$ such that $H' \cong H_1$ with $\langle S \cup \{w\} \rangle$ containing H' containing w. Note that $\langle S - \{v\} \cup \{w\} \rangle$ does not contain any subgraph H'' containing w such that $H'' \cong H_2$ for any $H_2 \in \mathscr{H}$. Furthermore, $V(H') \cap (S' - \{w\}) = \emptyset$, so that $H' \in p\mathscr{H}\ (w,S')$. Also, $p\mathscr{H}\ (w, S' \cup \{v\}) = \emptyset$. Suppose not. Then there exists $H'' \subseteq G$ and $H_2 \in \mathscr{H}$ such that $H'' \cong H_2$, $w \in V(H'')$

and $V(H'')\cap(S'\cup\{v\}-\{w\})=\emptyset$. This implies that $V(H'')\subseteq\langle S-\{v\}\cup\{w\}\rangle$, which is a contradiction.

Conversely, suppose that S is a $v\mathscr{H}$-external redundant set. We show that $S'=V-S$ is an $\mathscr{H}$-irredundant set. Suppose not. Then there exists $v\in S'$ such that $\mathrm{PN}_{\mathscr{H}}[v,S']=\emptyset$. If v is not $\mathscr{H}$-adjacent in S' to any other vertex of S', then $\mathrm{PN}_{\mathscr{H}}[v,S']\neq\emptyset$, a contradiction. This means that there exists $H'\subseteq G$ and $H\in\mathscr{H}$ such that $H'\cong H$, $v\in V(H')$ and $V(H')\subseteq S'$. But then $V(H')\cap(S\cup\{v\}-\{v\})=\emptyset$, whence $H'\in p\mathscr{H}\ (v,S\cup\{v\})$. Since S is a $v\mathscr{H}$-external redundant set, there is a $w\in S$ such that $p\mathscr{H}\ (w,S)\neq\emptyset$ and $p\mathscr{H}\ (w,S\cup\{v\})=\emptyset$. Let $H'\in p\mathscr{H}\ (w,S)$ where $H'\cong H$ for some $H\in\mathscr{H}$. Then $w\in V(H')$ and $V(H')\cap(S-\{w\})=\emptyset$. Also, since $p\mathscr{H}\ (w,S\cup\{v\})=\emptyset$, it follows that $V(H')\cap(S\cup\{v\}-\{w\})=\{v\}$, so that $v\in V(H')$, while $\langle S'\cup\{w\}\rangle$ contains H' containing w. This implies that $w\in N_{\mathscr{H}}^{S'}[v]$. Also, $w\notin N_{\mathscr{H}}^{S'-\{v\}}[S'-\{v\}]$: if this is the case, then there exists $H''\subseteq G$ and $H_2\in\mathscr{H}$ such that $H''\cong H_2$ with $\langle S'-\{v\}\cup\{w\}\rangle$ containing H'' containing w. But then $V(H'')\cap(S\cup\{v\}-\{w\})=\emptyset$, so that $H''\in p\mathscr{H}\ (w,S\cup\{v\})$, which is a contradiction. It therefore follows that $w\in\mathrm{PN}_{\mathscr{H}}[v,S']$. This final contradiction establishes our result. □

Families $\mathscr{T}_1$ and $\mathscr{T}_2$ of subsets of V are *complement related* when $X\in\mathscr{T}_1$ if and only if $V-X\in\mathscr{T}_2$. If $\mathscr{T}$ is any family of subsets of V, let $m(\mathscr{T})=\min\{|X|\mid X\in\mathscr{T}\}$ and $M(\mathscr{T})=\max\{|X|\mid X\in\mathscr{T}\}$. Further, let $\mathscr{T}^+$ denote the family of those members of $\mathscr{T}$ which are set-theoretically maximal with respect to membership, and $\mathscr{T}^-$ those who are minimal. The following result appears in [43].

Theorem 8. *If $\mathscr{T}_1$ and $\mathscr{T}_2$ are complement related families of subsets of V, then $M(\mathscr{T}_1)+m(\mathscr{T}_2)=p(G)=m(\mathscr{T}_1^+)+M(\mathscr{T}_2^-)$.*

We are now in a position to present our generalized Gallai theorems.

Theorem 9. *Let G be a graph and $\mathscr{H}$ be a set of nontrivial subgraphs of G. Then*

$$\mathrm{ir}_{\mathscr{H}}(G)+\mathrm{ER}'_{\mathscr{H}}(G)=p(G), \tag{9}$$

$$\gamma_{\mathscr{H}}(G)+\mathrm{IR}'_{\mathscr{H}}(G)=p(G), \tag{10}$$

$$i_{\mathscr{H}}(G)+\Gamma'_{\mathscr{H}}(G)=p(G), \tag{11}$$

$$\beta_{\mathscr{H}}(G)+\gamma'_{\mathscr{H}}(G)=p(G), \tag{12}$$

$$\Gamma_{\mathscr{H}}(G)+\mathrm{ir}'_{\mathscr{H}}(G)=p(G), \tag{13}$$

$$\mathrm{IR}_{\mathscr{H}}(G)+\mathrm{er}'_{\mathscr{H}}(G)=p(G). \tag{14}$$

Proof. Statements (11) and (12) follow from Lemma 1 and Theorem 8. Statements (10) and (13) follow from Lemma 2 and Theorem 8. Statements (9) and (14) follow from Lemma 3 and Theorem 8. □

Notice that statement (12) generalizes the result of Gallai (cf. Theorem 3), statement (11) generalizes the result of McFall and Nowakowski (cf. Theorem 4), statement (10) generalizes the result of Nieminen (cf. Theorem 5), while statement (13) generalizes the result of Slater (cf. Theorem 6). Statements (9) and (14) are new, even when restricted to $\mathscr{H}=\{K_2\}$, in which case $\mathrm{ir}_{\mathscr{H}}=\mathrm{ir}$ and $\mathrm{IR}_{\mathscr{H}}=\mathrm{IR}$.

4. Set systems

In Section 2 it was shown that certain hereditary families of vertex subsets of graphs could be used to define inequality chains. We now indicate a further generalization and use hereditary families of subsets of an arbitrary set to generate such chains.

Let $\mathscr{T}$ denote a family of subsets of a given set X and let $\overline{\mathscr{T}}=\mathscr{P}(X)-\mathscr{T}$. Elements of $\mathscr{T}$ are called *$\mathscr{T}$-sets.* $T\in\mathscr{T}$ is a *minimal* (*maximal*), respectively) *$\mathscr{T}$-set* if no proper subset (superset, respectively) of T is in $\mathscr{T}$. Let $\mathscr{T}_1$ be a family of subsets of X which is hereditary (to help the intuition, think of $\mathscr{T}_1$ as the family of independent sets of a graph G with $X=V(G)$). For $T\subseteq X$, we say that T *$\mathscr{T}_1$-covers* $x\in X$ if and only if $x\in T$ or there exists a minimal $\overline{\mathscr{T}_1}$-set, say Z, such that $x\in Z\subseteq T\cup\{x\}$. Let $\mathscr{T}_2=\{S\subseteq X|$ for all $x\in X-S$, $S\mathscr{T}_1$-covers $x.\}$ (think: dominating sets of G). For $u\in S\subseteq X$, let $\mathrm{PN}[u,S]=\{w\in X-(S-\{u\})\,|\,S\mathscr{T}_1\text{-covers } w \text{ and } S-\{u\} \text{ does not } \mathscr{T}_1\text{-cover } w\}$. Let $\mathscr{T}_3=\{S\subseteq X|$ for all $u\in S$, $\mathrm{PN}[u,S]\neq\emptyset\}$ (think: irredundant sets of G). Finally, let $\mathscr{T}_4=\{S\subseteq X|$ for all $S'\supset S$ there exists $u\in S'$ such that $\mathrm{PN}[u,S']=\emptyset$ and if $u\in S$, then $\mathrm{PN}[u,S]\neq\emptyset\}$ (think: external redundant sets of G).

Proposition 11. *$\mathscr{T}_2$ is superhereditary.*

Proof. Let $S\in\mathscr{T}_2$ and $S\subseteq S'$. We will show that $S'\in\mathscr{T}_2$. Let $x\in X-S'$. Then $x\in X-S$ and so $S\mathscr{T}_1$-covers x. It follows that $S'\mathscr{T}_1$-covers x and so $S'\in\mathscr{T}_2$. □

Proposition 12. *If each minimal $\overline{\mathscr{T}_1}$ has cardinality two, then $\mathscr{T}_3$ is hereditary.*

Proof. Let $S\in\mathscr{T}_3$ and let $S'\subseteq S$. We will show that $S'\in\mathscr{T}_3$. Suppose $u\in S'$. Then $u\in S$ and since $S\in\mathscr{T}_3$, $\mathrm{PN}[u,S]\neq\emptyset$. Let $w\in\mathrm{PN}[u,S]$. Then $w\in S-\{u\}$ and there are two cases to consider.

Case 1. $w=u$. Then $w\in S'$ and so $S'\mathscr{T}_1$-covers w. If $S'-\{u\}\,\mathscr{T}_1$-covers w, then $S-\{u\}\,\mathscr{T}_1$-covers w, contradicting the fact that $w\in\mathrm{PN}[u,S]$.

Case 2. $w\neq u$. Then $w\in X-S$. Since $S\mathscr{T}_1$-covers $w, S-\{u\}$ does not and minimal $\overline{\mathscr{T}_1}$-sets have size two, $\{u,v\}$ is a minimal $\overline{\mathscr{T}_1}$-set and so $S'\mathscr{T}_1$-covers w. Further, if $S'-\{u\}\,\mathscr{T}_1$-covers w, then so does $S-\{u\}$, a contradiction. We have proved that $S'\in\mathscr{T}_3$, whence $\mathscr{T}_3$ is hereditary. □

Proposition 13. $\mathscr{T}_1\subseteq\mathscr{T}_3$.

Proof. Let $S \in \mathscr{T}_1$ and $u \in S$. We will show that $u \in \mathrm{PN}[u,S]$. Firstly, $S\mathscr{T}_1$-covers u. Secondly, if $S - \{u\}\mathscr{T}_1$-covers u, then there exists a minimal $\overline{\mathscr{T}_1}$-set Y such that $u \in Y \subseteq (S - \{u\}) \cup \{u\} = S$. This implies, since $\mathscr{T}_1$ is hereditary, that $Y \in \mathscr{T}_1$, which is a contradiction. $\square$

Proposition 14. *If S is a maximal $\mathscr{T}_1$-set, then S is a minimal $\mathscr{T}_2$-set.*

Proof. Let S be a maximal $\mathscr{T}_1$-set. We first show that $S \in \mathscr{T}_2$. Let $x \in X - S$. Then $S \cup \{x\}$ is a $\overline{\mathscr{T}_1}$-set and contains a minimal $\overline{\mathscr{T}_1}$-set Z. Since S contains no $\overline{\mathscr{T}_1}$-set, we have that $x \in Z \subseteq S \cup \{x\}$ and so $S\mathscr{T}_1$-covers x. Hence $S \in \mathscr{T}_2$.

We now show that S is a *minimal* $\mathscr{T}_2$-set. Suppose, to the contrary, that some proper subset S' of S is in $\mathscr{T}_2$. Let $y \in S - S'$. Then, by the definition of $\mathscr{T}_2$, $S'\mathscr{T}_1$-covers y, i.e., there exists a minimal $\overline{\mathscr{T}_1}$-set Z with $y \in Z \subseteq S' \cup \{y\} \subseteq S$. Hence, $Z \in \mathscr{T}_1$, which is a contradiction. $\square$

Proposition 15. *If S is a minimal $\mathscr{T}_2$-set, then S is a maximal $\mathscr{T}_3$-set.*

Proof. Let S be a minimal $\mathscr{T}_2$-set. If $S \notin \mathscr{T}_3$, there exists $u \in S$ such that $\mathrm{PN}[u,S] = \emptyset$. Since $S\mathscr{T}_1$-covers $w \in X - (S - \{u\})$ and $\mathrm{PN}[u,S] = \emptyset$, $S - \{u\}\,\mathscr{T}_1$-covers every such vertex w. Hence $S - \{u\} \in \mathscr{T}_2$, which is a contradiction. We now show that S is a *maximal* $\mathscr{T}_3$-set. Suppose, to the contrary, that there exists $S' \supset S$ such that $S' \in \mathscr{T}_3$. Let $x \in S' - S$. Since $S \in \mathscr{T}_2$, there exists $Z \in \overline{\mathscr{T}_1}$ such that $x \in Z \subseteq S \cup \{x\} \subseteq S'$. Since $S' \in \mathscr{T}_3$, there exists $w \in X - (S' - \{x\})$ such that $S'\mathscr{T}_1$-covers w, but $S' - \{x\}$ does not $\mathscr{T}_1$-cover w. This implies that S does not $\mathscr{T}_1$-cover w, which contradicts the fact that $S \in \mathscr{T}_2$. $\square$

Proposition 16. *If S is a maximal $\mathscr{T}_3$-set, then S is a minimal $\mathscr{T}_4$-set.*

Proof. Let S be a maximal $\mathscr{T}_3$-set. Let $S' \supset S$. Then, since S is a maximal $\mathscr{T}_3$-set, $S' \notin \mathscr{T}_3$. Hence, there exists $u \in S'$ such that $\mathrm{PN}[u,S'] = \emptyset$ and if $u \in S$, then $\mathrm{PN}[u,S] \neq \emptyset$. Hence, $S \in \mathscr{T}_4$. We now prove that S is a *minimal* $\mathscr{T}_4$-set. Suppose not. Then there exists $S' \subset S$ such that $S' \in \mathscr{T}_4$. Since $S' \in \mathscr{T}_4$ and $S \supset S'$, there exists $u \in S$ such that $\mathrm{PN}[u,S] = \emptyset$ and if $u \in S'$, then $\mathrm{PN}[u,S'] \neq \emptyset$. Since $S \in \mathscr{T}_3$, $\mathrm{PN}[u,S] \neq \emptyset$, which is a contradiction. $\square$

Let $\pi_1(\Pi_1), \pi_2(\Pi_2), \pi_3(\Pi_3), \pi_4(\Pi_4)$ be the minimum (maximum) cardinality of a maximal $\mathscr{T}_1$-set, a minimal $\mathscr{T}_2$-set, a maximal $\mathscr{T}_3$-set and a minimal $\mathscr{T}_4$-set. We have the following inequality chain.

Theorem 10. $\pi_4 \leqslant \pi_3 \leqslant \pi_2 \leqslant \pi_1 \leqslant \Pi_1 \leqslant \Pi_2 \leqslant \Pi_3 \leqslant \Pi_4$.

Proof. Immediate from Propositions 14–16. $\square$

Proposition 17. *S is a maximal $\mathscr{T}_1$-set if and only if $S \in \mathscr{T}_1 \cap \mathscr{T}_2$.*

Proof. If S is a maximal $\mathscr{T}_1$-set, the proof of Proposition 14 shows that S is a $\mathscr{T}_2$-set. Hence $S \in \mathscr{T}_1 \cap \mathscr{T}_2$.

Conversely, let $S \in \mathscr{T}_1 \cap \mathscr{T}_2$. We show that S is a *maximal* $\mathscr{T}_1$-set. Let $S' \supset S$ and let $x \in S' - S$. Since $S \in \mathscr{T}_2$, S $\mathscr{T}_1$-covers x. Hence, there exists a minimal $\overline{\mathscr{T}_1}$-set, Z, such that $z \in Z \subseteq S \cup \{x\} \subseteq S'$. By the hereditary property of $\mathscr{T}_1$, $S' \notin \mathscr{T}_1$. □

Proposition 18. *S is a minimal $\mathscr{T}_2$-set if and only if $S \in \mathscr{T}_2 \cap \mathscr{T}_3$.*

Proof. If S is a minimal $\mathscr{T}_2$-set, the proof of Proposition 15 shows that S is a $\mathscr{T}_3$-set. Hence $S \in \mathscr{T}_2 \cap \mathscr{T}_3$.

Conversely, let $S \in \mathscr{T}_2 \cap \mathscr{T}_3$. We show that S is a *minimal* $\mathscr{T}_2$-set. Let $u \in S$. Then, by the definition of $\mathscr{T}_3$, there exists $w \in X - (S - \{u\})$ such that $S - \{u\}$ does not $\mathscr{T}_1$-cover w. Hence, $S - \{u\} \notin \mathscr{T}_2$ and S is a minimal $\mathscr{T}_2$-set. □

Proposition 19. *S is a maximal $\mathscr{T}_3$-set if and only if $S \in \mathscr{T}_3 \cap \mathscr{T}_4$.*

Proof. If S is a maximal $\mathscr{T}_3$-set, the proof of Proposition 16 shows that S is a $\mathscr{T}_4$-set.

Conversely, let $S \in \mathscr{T}_3 \cap \mathscr{T}_4$ and let S' be any proper superset of S. Then, since $S \in \mathscr{T}_4$, there exists $u \in S'$ such that $\mathrm{PN}[u, S'] = \emptyset$. This shows that $S' \notin \mathscr{T}_3$, so that S is a maximal $\mathscr{T}_3$-set. □

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